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Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type

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Abstract

We prove a common fixed point theorem of Gregus type for four mappings satisfying a contractive condition of integral type in metric spaces using the concept of weak compatibility which generalizes Theorem 2 of [A. Djoudi, L. Nisse, Gregus type fixed points for weakly compatible mappings, *Bull. Belg. Math. Soc.* 10 (2003) 369–378] and other papers. We prove also common fixed point theorems of Gregus type using a strict contractive condition of integral type, a property (E.A) and a common property (E.A) introduced by [M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.* 270 (2002) 181–188] and [W. Liu, J. Wu, Z. Li, Common fixed points of single-valued and multi-valued maps, *Int. J. Math. Math. Sci.* 19 (2005) 3045–3055], respectively.

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1. Introduction

Let S and T be self-mappings of a metric space (X, d) . S and T are commuting if $STx = T Sx$ for all $x \in X$. Sessa [17] defined S and T to be weakly commuting if for all $x \in X$,

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$$d(STx, TSx) \leq d(Tx, Sx). \quad (1.1)$$

Jungck [5] defined S and T to be compatible as a generalization of weakly commuting if

$$\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0, \quad (1.2)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper. See [5] and [17].

Jungck et al. [6] defined S and T to be compatible mappings of type (A) if

$$\lim_{n \rightarrow \infty} d(STx_n, T^2x_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(TSx_n, S^2x_n) = 0, \quad (1.3)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

Clearly, weakly commuting implies compatible of type (A). By [6], the converse is not true. Examples are given to show that the two concepts of compatibility are independent. See [6].

Recently, Pathak and Khan [12] defined S and T to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\begin{aligned} \lim_{n \rightarrow \infty} d(TSx_n, S^2x_n) &\leq \frac{1}{2} \left[\lim_{n \rightarrow \infty} d(TSx_n, Tt) + \lim_{n \rightarrow \infty} d(Tt, T^2x_n) \right] \quad \text{and} \\ \lim_{n \rightarrow \infty} d(STx_n, T^2x_n) &\leq \frac{1}{2} \left[\lim_{n \rightarrow \infty} d(STx_n, St) + \lim_{n \rightarrow \infty} d(St, S^2x_n) \right], \end{aligned} \quad (1.4)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true. See [11]. However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if S and T are continuous. See [12].

Pathak et al. [13] defined S and T to be compatible mappings of type (P) if

$$\lim_{n \rightarrow \infty} d(S^2x_n, T^2x_n) = 0, \quad (1.5)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if S and T are continuous. See [13].

Pathak et al. [14] defined S and T to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

$$\begin{aligned} \lim_{n \rightarrow \infty} d(TSx_n, S^2x_n) &\leq \frac{1}{3} \left[\lim_{n \rightarrow \infty} d(TSx_n, Tt) + \lim_{n \rightarrow \infty} d(Tt, S^2x_n) + \lim_{n \rightarrow \infty} d(Tt, T^2x_n) \right], \\ \lim_{n \rightarrow \infty} d(STx_n, T^2x_n) &\leq \frac{1}{3} \left[\lim_{n \rightarrow \infty} d(STx_n, St) + \lim_{n \rightarrow \infty} d(St, T^2x_n) + \lim_{n \rightarrow \infty} d(St, S^2x_n) \right], \end{aligned} \quad (1.6)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous. See [14].

2. Preliminaries

Definition 1. Jungck and Rhoades [7] defined S and T to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

Lemma 1. [5,6,12–14] *If S and T are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible.*

The following example shows that the converse is not true in general.

Example. Let $(X, d) = ([0, 10], | \cdot |)$. Define S and T by

$$Sx = \begin{cases} 3 & \text{if } x \in (0, 2], \\ 0 & \text{if } x \in \{0\} \cup (2, 10], \end{cases} \quad Tx = \begin{cases} 0 & \text{if } x = 0, \\ x + 8 & \text{if } x \in (0, 2], \\ x - 2 & \text{if } x \in (2, 10]. \end{cases}$$

We have $Sx = Tx$ iff $x = 0$. $ST(0) = T(0) = 0$, $TS(0) = S(0) = 0$. Then, $\{S, T\}$ is weakly compatible. Let $\{x_n\}$ be a sequence in X defined by: $x_n = 2 + \frac{1}{n}$, $n \geq 1$. $Sx_n = S(2 + \frac{1}{n}) = 0$, $Tx_n = T(2 + \frac{1}{n}) = \frac{1}{n}$. $Sx_n, Tx_n \rightarrow t = 0$ as $n \rightarrow \infty$. $STx_n = S(\frac{1}{n}) = 3$, $TSx_n = T(0) = 0$, $|STx_n - TSx_n| \rightarrow 3 \neq 0$. So, $\{S, T\}$ is not compatible.

$S^2x_n = S(0) = 0$, $T^2x_n = T(\frac{1}{n}) = 8 + \frac{1}{n}$. Therefore, $|TSx_n - S^2x_n| = 0$ and $|STx_n - T^2x_n| = 5 + \frac{1}{n} \rightarrow 5 \neq 0$. Then, $\{S, T\}$ is not compatible of type (A).

$$\lim_{n \rightarrow \infty} |STx_n - T^2x_n| = 5 > \frac{1}{2} \left[\lim_{n \rightarrow \infty} |STx_n - St| + \lim_{n \rightarrow \infty} |St - S^2x_n| \right] = \frac{3}{2}.$$

Hence, $\{S, T\}$ is not compatible of type (B). $\lim_{n \rightarrow \infty} |S^2x_n - T^2x_n| = 8 \neq 0$. Therefore, $\{S, T\}$ is not compatible of type (P).

$$\begin{aligned} \lim_{n \rightarrow \infty} |STx_n - T^2x_n| &= 5 > \frac{1}{3} \left[\lim_{n \rightarrow \infty} |STx_n - St| + \lim_{n \rightarrow \infty} |St - T^2x_n| + \lim_{n \rightarrow \infty} |St - S^2x_n| \right] \\ &= \frac{11}{3}. \end{aligned}$$

So, $\{S, T\}$ is not compatible of type (C).

Definition 2. Pant [10] defined S and T to be R -weakly commuting at a point $x \in X$ if for some $R > 0$,

$$d(STx, TSx) \leq Rd(Tx, Sx). \quad (2.1)$$

Definition 3. S and T are pointwise R -weakly commuting on X if given $x \in X$, there exists $R > 0$ such that (2.1) holds.

It is proved in [11] that R -weak commutativity is equivalent to commutativity at coincidence points; i.e., S and T are pointwise R -weakly commuting if and only if they are weakly compatible.

M. Aamri and D. El Moutawakil [1] defined property (E.A) as follows.

Definition 4. Let $S, T : X \rightarrow X$. The pair (S, T) satisfies property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t \in X. \quad (2.2)$$

It is clear from the definition of compatibility that the pair (S, T) of a metric space (X, d) is noncompatible if there exists at least one sequence $\{x_n\}$ in X such that (2.2) holds but,

$\lim_{n \rightarrow \infty} d(STx_n, TSx_n)$ is either nonzero or does not exist. Therefore, two noncompatible mappings of a metric space (X, d) satisfy property (E.A).

Recently, Y. Liu et al. [9] defined a common property (E.A) as follows.

Definition 5. Let $A, S, B, T : X \rightarrow X$. The pairs (A, S) and (B, T) satisfy a common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = t \in X. \quad (2.3)$$

If $B = A$ and $T = S$ in (2.3), we obtain the definition of property (E.A).

Gregus [4] proved the following theorem.

Theorem 1. Let C be a nonempty closed convex subset of a Banach space X and T be a mapping of C into itself satisfying the inequality

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

for all x, y in C , where $a > 0$, $b, c \geq 0$, $a + b + c = 1$. Then, T has a unique fixed point.

Several authors have generalized Theorem 1. See [3,8,12,14].

Let \mathbb{R}_+ be the set of nonnegative real numbers and F the family of mappings φ from \mathbb{R}_+ into \mathbb{R}_+ such that each φ is upper semicontinuous, nondecreasing and $\varphi(t) < t$ for all $t > 0$.

The following theorem has been proved in [3].

Theorem 2. Let A, B, S and T be mappings from a Banach space X into itself satisfying

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X),$$

$$\begin{aligned} \|Ax - By\|^p \leq & \varphi \left(a\|Sx - Ty\|^p + (1-a) \max \left\{ \alpha\|Ax - Sx\|^p, \beta\|By - Ty\|^p, \right. \right. \\ & \|Ax - Sx\|^{\frac{p}{2}} \cdot \|Ax - Ty\|^{\frac{p}{2}}, \|Ax - Ty\|^{\frac{p}{2}} \cdot \|Sx - By\|^{\frac{p}{2}}, \\ & \left. \left. \frac{1}{2}(\|Ax - Sx\|^p + \|By - Ty\|^p) \right\} \right) \end{aligned} \quad (2.4)$$

for all x, y in X , where $0 < a \leq 1$, $0 < \alpha, \beta \leq 1$, $p \geq 1$ and $\varphi \in F$. If $A(X)$ or $B(X)$ is closed and the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X .

Since for any two real nonnegative numbers b and c , $\frac{b+c}{2} \leq \max\{b, c\}$ and φ is nondecreasing, therefore the right-hand side of (2.4) implies the following inequality:

$$\begin{aligned} \|Ax - By\|^p \leq & \varphi \left(a\|Sx - Ty\|^p + (1-a) \max \{ \|Ax - Sx\|, \|By - Ty\|, \right. \\ & \|Ax - Sx\|^{\frac{1}{2}} \cdot \|Ax - Ty\|^{\frac{1}{2}}, \|Sx - By\|^{\frac{1}{2}} \cdot \|Ax - Ty\|^{\frac{1}{2}} \}^p \Big). \end{aligned} \quad (2.5)$$

Lemma 2. [18] For any $t > 0$, $\varphi(t) < t$ iff $\lim_{n \rightarrow +\infty} \varphi^n(t) = 0$, where φ^n denotes the n -times repeated composition of φ with itself.

3. Main results

Let A, B, S and T be mappings from a metric space (X, d) into itself such that

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \quad (3.1)$$

$$\begin{aligned} \left(\int_0^{d(Ax, By)} \psi(t) dt \right)^p &\leq \varphi \left[a \left(\int_0^{d(Sx, Ty)} \psi(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(Ax, Sx)} \psi(t) dt, \right. \right. \\ &\quad \int_0^{d(By, Ty)} \psi(t) dt, \left(\int_0^{d(Ax, Sx)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt \right)^{\frac{1}{2}}, \\ &\quad \left. \left(\int_0^{d(Sx, By)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt \right)^{\frac{1}{2}} \right\}^p \Big] \end{aligned} \quad (3.2)$$

for all x, y in X , where $0 < a \leq 1$, $p \geq 1$ and $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Lebesgue integrable mapping which is summable nonnegative and such that

$$\int_0^\epsilon \psi(t) dt > 0 \quad \text{for each } \epsilon > 0. \quad (3.3)$$

By (3.1), for an arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$, for this point x_1 we can choose a point x_2 such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad (3.4)$$

for all $n = 0, 1, 2, \dots$

Lemma 3. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying (3.1) and (3.2) for all x, y in X , where $0 < a \leq 1$, $p \geq 1$ and ψ satisfies (3.3). Then, the sequence $\{y_n\}$ defined by (3.4) is a Cauchy sequence in X .

Proof. First of all, assume that $y_n \neq y_{n+1}$ for all n .

By (3.2) and (3.4) we have

$$\begin{aligned} \left(\int_0^{d(y_{2n}, y_{2n+1})} \psi(t) dt \right)^p &= \left(\int_0^{d(Ax_{2n}, Bx_{2n+1})} \psi(t) dt \right)^p \\ &\leq \varphi \left[a \left(\int_0^{d(y_{2n-1}, y_{2n})} \psi(t) dt \right)^p \right. \\ &\quad \left. + (1-a) \max \left\{ \int_0^{d(y_{2n-1}, y_{2n})} \psi(t) dt, \int_0^{d(y_{2n+1}, y_{2n})} \psi(t) dt \right\}^p \right]. \end{aligned}$$

If

$$\int_0^{d(y_{2n-1}, y_{2n})} \psi(t) dt \leq \int_0^{d(y_{2n+1}, y_{2n})} \psi(t) dt$$

in the above inequality, then

$$\left(\int_0^{d(y_{2n+1}, y_{2n})} \psi(t) dt \right)^p \leq \varphi \left(\left(\int_0^{d(y_{2n+1}, y_{2n})} \psi(t) dt \right)^p \right) < \left(\int_0^{d(y_{2n+1}, y_{2n})} \psi(t) dt \right)^p$$

which is a contradiction. Therefore

$$\left(\int_0^{d(y_{2n}, y_{2n+1})} \psi(t) dt \right)^p \leq \varphi \left(\left(\int_0^{d(y_{2n-1}, y_{2n})} \psi(t) dt \right)^p \right).$$

Similarly, we get

$$\left(\int_0^{d(y_{2n+1}, y_{2n+2})} \psi(t) dt \right)^p \leq \varphi \left(\left(\int_0^{d(y_{2n}, y_{2n+1})} \psi(t) dt \right)^p \right).$$

By induction we obtain

$$\left(\int_0^{d(y_n, y_{n+1})} \psi(t) dt \right)^p \leq \varphi \left(\left(\int_0^{d(y_{n-1}, y_n)} \psi(t) dt \right)^p \right) \leq \dots \leq \varphi^n \left(\left(\int_0^{d(y_0, y_1)} \psi(t) dt \right)^p \right). \quad (3.5)$$

By Lemma 2, it follows that

$$\lim_{n \rightarrow \infty} \int_0^{d(y_n, y_{n+1})} \psi(t) dt = 0 \quad (3.6)$$

and (3.3) implies that

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \quad (3.7)$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in X . By (3.7), it suffices to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X . Suppose not. As in [3] we have

$$d(y_{2n(k)}, y_{2m(k)}) \rightarrow \varepsilon \quad \text{as } k \rightarrow \infty, \quad (3.8)$$

$$d(y_{2n(k)}, y_{2m(k)-1}) \rightarrow \varepsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \rightarrow \varepsilon \quad \text{as } k \rightarrow \infty. \quad (3.9)$$

Using (3.4) we have

$$d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2m(k)+1}).$$

By (3.6) and (3.2) we get

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} \int_0^{d(y_{2n(k)}, y_{2m(k)})} \psi(t) dt \\
 & \leq \lim_{k \rightarrow \infty} \int_0^{d(Ax_{2m(k)}, Bx_{2n(k)+1})} \psi(t) dt \\
 & \leq \lim_{k \rightarrow \infty} \left(\varphi \left[a \left(\int_0^{d(Sx_{2m(k)}, Tx_{2n(k)+1})} \psi(t) dt \right)^p \right. \right. \\
 & \quad \left. \left. + (1-a) \max \left\{ \int_0^{d(Ax_{2m(k)}, Sx_{2m(k)})} \psi(t) dt, \int_0^{d(Bx_{2n(k)+1}, Tx_{2n(k)+1})} \psi(t) dt, \right. \right. \right. \\
 & \quad \left. \left(\int_0^{d(Ax_{2m(k)}, Sx_{2m(k)})} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax_{2m(k)}, Tx_{2n(k)+1})} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\
 & \quad \left. \left(\int_0^{d(Sx_{2m(k)}, Bx_{2n(k)+1})} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax_{2m(k)}, Tx_{2n(k)+1})} \psi(t) dt \right)^{\frac{1}{2}} \right]^p \right]^{\frac{1}{p}} \\
 & = \lim_{k \rightarrow \infty} \left(\varphi \left[a \left(\int_0^{d(y_{2m(k)-1}, y_{2n(k)})} \psi(t) dt \right)^p \right. \right. \\
 & \quad \left. \left. + (1-a) \max \left\{ \int_0^{d(y_{2m(k)}, y_{2m(k)-1})} \psi(t) dt, \int_0^{d(y_{2n(k)+1}, y_{2n(k)})} \psi(t) dt, \right. \right. \right. \\
 & \quad \left. \left(\int_0^{d(y_{2m(k)}, y_{2m(k)-1})} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(y_{2n(k)+1}, y_{2n(k)})} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\
 & \quad \left. \left(\int_0^{d(y_{2m(k)-1}, y_{2n(k)+1})} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(y_{2m(k)}, y_{2n(k)})} \psi(t) dt \right)^{\frac{1}{2}} \right]^p \right]^{\frac{1}{p}}.
 \end{aligned}$$

Using (3.6), (3.8) and (3.9) we have as $k \rightarrow \infty$

$$\int_0^\epsilon \psi(t) dt \leq \left[\varphi \left(a \left(\int_0^\epsilon \psi(t) dt \right)^p + (1-a) \left(\int_0^\epsilon \psi(t) dt \right)^p \right) \right]^{\frac{1}{p}} < \int_0^\epsilon \psi(t) dt$$

which is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in X . \square

Theorem 3. Let A , B , S and T be mappings from a metric space (X, d) into itself satisfying (3.1), (3.2) for all x, y in X , where $0 < a \leq 1$, $p \geq 1$ and ψ satisfies (3.3). Suppose that one of $S(X)$ or $T(X)$ is complete and the pairs (A, S) and (B, T) are weakly compatible. Then, A , B , S and T have a unique common fixed point in X .

Proof. By Lemma 3, the sequence $\{y_{2n+1}\} = \{Sx_{2n+2}\} \subset S(X)$ is a Cauchy sequence in $S(X)$. Since $S(X)$ is complete, it converges to a point $z = Su$ for some $u \in X$. Hence, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ also converge to z .

If $Au \neq z$, using (3.2) we get

$$\begin{aligned} \left(\int_0^{d(Au, Bx_{2n+1})} \psi(t) dt \right)^p &\leq \varphi \left[a \left(\int_0^{d(Su, Tx_{2n+1})} \psi(t) dt \right)^p \right. \\ &\quad \left. + (1-a) \max \left\{ \int_0^{d(Au, Su)} \psi(t) dt, \int_0^{d(Bx_{2n+1}, Tx_{2n+1})} \psi(t) dt, \right. \right. \\ &\quad \left. \left(\int_0^{d(Au, Su)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tx_{2n+1})} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\ &\quad \left. \left(\int_0^{d(Su, Bx_{2n+1})} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tx_{2n+1})} \psi(t) dt \right)^{\frac{1}{2}} \right\}^p \Big]. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain

$$\left(\int_0^{d(Au, z)} \psi(t) dt \right)^p \leq \varphi \left[(1-a) \left(\int_0^{d(Au, z)} \psi(t) dt \right)^p \right] < \left(\int_0^{d(Au, z)} \psi(t) dt \right)^p$$

which is a contradiction. Then

$$\int_0^{d(Au, z)} \psi(t) dt = 0,$$

and (3.3) implies that $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $z = Tv$.

If $z \neq Bv$, using (3.2) we have

$$\begin{aligned} \left(\int_0^{d(z, Bv)} \psi(t) dt \right)^p &= \left(\int_0^{d(Au, Bv)} \psi(t) dt \right)^p \\ &\leq \varphi \left[a \left(\int_0^{d(Su, Tv)} \psi(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(Au, Su)} \psi(t) dt, \right. \right. \\ &\quad \left. \int_0^{d(Bv, Tv)} \psi(t) dt, \left(\int_0^{d(Au, Su)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tv)} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\ &\quad \left. \left(\int_0^{d(Su, Bv)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tv)} \psi(t) dt \right)^{\frac{1}{2}} \right\}^p \Big] \end{aligned}$$

$$\begin{aligned}
&= \varphi \left[(1-a) \left(\int_0^{d(z, Bv)} \psi(t) dt \right)^p \right] \\
&< \left(\int_0^{d(z, Bv)} \psi(t) dt \right)^p
\end{aligned}$$

which is a contradiction. Therefore, $z = Bv = Tv$.

Since the pair (A, S) is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$.

If $Az \neq z$, using (3.2) we obtain

$$\begin{aligned}
\left(\int_0^{d(Az, z)} \psi(t) dt \right)^p &= \left(\int_0^{d(Az, Bv)} \psi(t) dt \right)^p \\
&\leq \varphi \left[a \left(\int_0^{d(Sz, Tv)} \psi(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(Az, Sz)} \psi(t) dt, \right. \right. \\
&\quad \left. \int_0^{d(Bv, Tv)} \psi(t) dt, \left(\int_0^{d(Az, Sz)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az, Tv)} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\
&\quad \left. \left(\int_0^{d(Sz, Bv)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az, Tv)} \psi(t) dt \right)^{\frac{1}{2}} \right]^p \\
&< \left(\int_0^{d(Az, z)} \psi(t) dt \right)^p
\end{aligned}$$

which is a contradiction. Hence, $z = Az = Sz$.

Similarly, we can prove that $z = Bz = Tz$.

Suppose there exists n such that $y_n = y_{n+1}$. Therefore, $y_n = y_{n+k}$ for $k \geq 1$. So, there exist $u, v \in X$ such that $Au = Su$ and $Bv = Tv$. As in Theorem 3, we can prove that $z = Az = Bz = Tz$.

The uniqueness of z follows from (3.2). \square

If $B = A$ and $T = S$ in Theorem 3, we get the following corollary.

Corollary 1. Let A and S be mappings from a metric space (X, d) into itself satisfying

$$A(X) \subset S(X), \quad (3.10)$$

$$\begin{aligned}
\left(\int_0^{d(Ax, Ay)} \psi(t) dt \right)^p &\leq \varphi \left[a \left(\int_0^{d(Sx, Sy)} \psi(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(Ax, Sx)} \psi(t) dt, \right. \right. \\
&\quad \left. \int_0^{d(Ay, Sy)} \psi(t) dt, \left(\int_0^{d(Ax, Sx)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Sy)} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\
&\quad \left. \left(\int_0^{d(Ax, Sy)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Sx)} \psi(t) dt \right)^{\frac{1}{2}} \right]^p
\end{aligned}$$

$$\left(\int_0^{d(Sx, Ay)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Sy)} \psi(t) dt \right)^{\frac{1}{2}} \Bigg\}^p \Bigg]$$

for all x, y in X , where $0 < a \leq 1$, $p \geq 1$ and ψ satisfies (3.3). Suppose that $S(X)$ is complete and the pair (A, S) is weakly compatible. Then, A and S have a unique common fixed point in X .

If $S = I_X$ in Corollary 1, where I_X is the identity mapping in X , we get the following corollary.

Corollary 2. Let A be a mapping from a Banach space (X, d) into itself satisfying

$$\begin{aligned} \left(\int_0^{d(Ax, Ay)} \psi(t) dt \right)^p &\leq \varphi \left[a \left(\int_0^{d(x, y)} \psi(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(x, Ax)} \psi(t) dt, \right. \right. \\ &\quad \int_0^{d(y, Ay)} \psi(t) dt, \left(\int_0^{d(x, Ax)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(y, Ay)} \psi(t) dt \right)^{\frac{1}{2}}, \\ &\quad \left. \left. \left(\int_0^{d(x, Ay)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(y, Ax)} \psi(t) dt \right)^{\frac{1}{2}} \right\}^p \right] \end{aligned}$$

for all x, y in X , where $0 < a \leq 1$, $p \geq 1$ and ψ satisfies (3.3). Then, A has a unique fixed point in X .

If $\psi(t) = 1$ in Theorem 3, we obtain Theorem 2 of [3].

If $p = a = 1$ and $\varphi(t) = kt$, $0 < k < 1$ in Corollary 2, we obtain Theorem 2.1 of [2].

Now, we prove a common fixed point Theorem of Gregus type using a strict contraction of integral type and property (E.A).

Theorem 4. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying (3.1) and

$$\begin{aligned} \int_0^{d(Ax, By)} \psi(t) dt &< a \int_0^{d(Sx, Ty)} \psi(t) dt + (1-a) \max \left\{ \int_0^{d(Ax, Sx)} \psi(t) dt, \right. \\ &\quad \int_0^{d(By, Ty)} \psi(t) dt, \left(\int_0^{d(Ax, Sx)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt \right)^{\frac{1}{2}}, \\ &\quad \left. \left(\int_0^{d(Sx, By)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Ty)} \psi(t) dt \right)^{\frac{1}{2}} \right\} \end{aligned} \quad (3.11)$$

for all x, y in X for which the right-hand side of (3.11) is positive, where $0 < a < 1$ and ψ satisfies (3.3). Suppose that (A, S) or (B, T) satisfies property (E.A), one of $A(X)$, $B(X)$, $S(X)$, $T(X)$ is a closed subspace of X and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X .

Proof. Suppose that (B, T) satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = z$ for some $z \in X$. Therefore, $\lim_{n \rightarrow \infty} d(Bx_n, Tx_n) = 0$. Since $B(X) \subset S(X)$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$.

Hence, $\lim_{n \rightarrow \infty} Sy_n = z$. Let us show that $\lim_{n \rightarrow \infty} Ay_n = z$.

Suppose that $\limsup_{n \rightarrow \infty} d(Ay_n, z) = \varepsilon > 0$. Using (3.11) we get

$$\begin{aligned} \int_0^{d(Ay_n, Bx_n)} \psi(t) dt &< a \int_0^{d(Sy_n, Tx_n)} \psi(t) dt + (1-a) \max \left\{ \int_0^{d(Ay_n, Sy_n)} \psi(t) dt, \right. \\ &\quad \int_0^{d(Bx_n, Tx_n)} \psi(t) dt, \left(\int_0^{d(Ay_n, Sy_n)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ay_n, Tx_n)} \psi(t) dt \right)^{\frac{1}{2}}, \\ &\quad \left. \left(\int_0^{d(Sy_n, Bx_n)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ay_n, Tx_n)} \psi(t) dt \right)^{\frac{1}{2}} \right\} \\ &= a \int_0^{d(Bx_n, Tx_n)} \psi(t) dt + (1-a) \max \left\{ \int_0^{d(Ay_n, Bx_n)} \psi(t) dt, \right. \\ &\quad \left. \int_0^{d(Bx_n, Tx_n)} \psi(t) dt, \left(\int_0^{d(Ay_n, Bx_n)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ay_n, Tx_n)} \psi(t) dt \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$\int_0^\varepsilon \psi(t) dt \leq (1-a) \int_0^\varepsilon \psi(t) dt < \int_0^\varepsilon \psi(t) dt$$

which is a contradiction. Hence

$$\int_0^\varepsilon \psi(t) dt = 0$$

and (3.3) implies that $\varepsilon = 0$; i.e., $\lim_{n \rightarrow \infty} Ay_n = z$.

Suppose that $S(X)$ is a closed subspace of X . Then, $z = Su$ for some $u \in X$.

If $Au \neq z$, using (3.11) we get

$$\begin{aligned} \int_0^{d(Au, Bx_{2n+1})} \psi(t) dt &< a \int_0^{d(Su, Tx_{2n+1})} \psi(t) dt \\ &\quad + (1-a) \max \left\{ \int_0^{d(Au, Su)} \psi(t) dt, \int_0^{d(Bx_{2n+1}, Tx_{2n+1})} \psi(t) dt, \right. \\ &\quad \left. \left(\int_0^{d(Au, Su)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tx_{2n+1})} \psi(t) dt \right)^{\frac{1}{2}} \right\}, \end{aligned}$$

$$\left(\int_0^{d(Su, Bx_{2n+1})} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tx_{2n+1})} \psi(t) dt \right)^{\frac{1}{2}} \Big\}.$$

Letting $n \rightarrow \infty$ we obtain

$$\int_0^{d(Au, z)} \psi(t) dt \leq (1-a) \int_0^{d(Au, z)} \psi(t) dt < \int_0^{d(Au, z)} \psi(t) dt$$

which is a contradiction. Then, $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $z = Tv$.

If $z \neq Bv$, using (3.11) we have

$$\begin{aligned} \int_0^{d(z, Bv)} \psi(t) dt &= \int_0^{d(Au, Bv)} \psi(t) dt \\ &< a \int_0^{d(Su, Tv)} \psi(t) dt + (1-a) \max \left\{ \int_0^{d(Au, Su)} \psi(t) dt, \right. \\ &\quad \left. \int_0^{d(Bv, Tv)} \psi(t) dt, \left(\int_0^{d(Au, Su)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tv)} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\ &\quad \left. \left(\int_0^{d(Su, Bv)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Tv)} \psi(t) dt \right)^{\frac{1}{2}} \right\} \\ &= (1-a) \int_0^{d(z, Bv)} \psi(t) dt \\ &< \int_0^{d(z-Bv)} \psi(t) dt \end{aligned}$$

which is a contradiction. Therefore, $z = Bv = Tv$.

Since the pair (A, S) is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$.

If $Az \neq z$, using (3.11) we obtain

$$\begin{aligned} \int_0^{d(Az, z)} \psi(t) dt &= \int_0^{d(Az, Bv)} \psi(t) dt \\ &< a \int_0^{d(Sz, Tv)} \psi(t) dt + (1-a) \max \left\{ \int_0^{d(Az, Sz)} \psi(t) dt, \right. \\ &\quad \left. \int_0^{d(Bv, Tv)} \psi(t) dt, \left(\int_0^{d(Az, Sz)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az, Tv)} \psi(t) dt \right)^{\frac{1}{2}}, \right. \\ &\quad \left. \left(\int_0^{d(Sz, Bv)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az, Tv)} \psi(t) dt \right)^{\frac{1}{2}} \right\} \end{aligned}$$

$$\left(\int_0^{d(Sz, Bv)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az, Tv)} \psi(t) dt \right)^{\frac{1}{2}} \Bigg\} \\ = \int_0^{d(Az, z)} \psi(t) dt$$

which is a contradiction. Hence, $z = Az = Sz$.

Similarly, we can prove that $z = Bz = Tz$. The uniqueness of z follows from (3.11). \square

If we let $B = A$ and $T = S$ in Theorem 4, we get the following corollary.

Corollary 3. *Let A and S be mappings from a metric space (X, d) into itself satisfying (3.10) and*

$$\int_0^{d(Ax, Ay)} \psi(t) dt < a \int_0^{d(Sx, Sy)} \psi(t) dt + (1-a) \max \left\{ \int_0^{d(Ax, Sx)} \psi(t) dt, \right. \\ \int_0^{d(Ay, Sy)} \psi(t) dt, \left(\int_0^{d(Ax, Sx)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Sy)} \psi(t) dt \right)^{\frac{1}{2}}, \\ \left. \left(\int_0^{d(Sx, Ay)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax, Sy)} \psi(t) dt \right)^{\frac{1}{2}} \right\} \quad (3.12)$$

for all x, y in X for which the right-hand side of (3.12) is positive, where $0 < a < 1$ and ψ satisfies (3.3). Suppose that (A, S) satisfies property (E.A), $A(X)$ or $S(X)$ is a closed subspace of X and the pair (A, S) is weakly compatible. Then, A and S have a unique common fixed point in X .

If $\psi(t) = 1$ in Theorem 4, we get the following corollary.

Corollary 4. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying (3.1) and*

$$d(Ax, By) < ad(Sx, Ty) + (1-a) \max \{ d(Ax, Sx), d(By, Ty), \\ d(Ax, Sx)^{\frac{1}{2}} \cdot d(Ax, Ty)^{\frac{1}{2}}, d(Ax, Ty)^{\frac{1}{2}} \cdot d(Sx, By)^{\frac{1}{2}} \} \quad (3.13)$$

for all x, y in X for which the right-hand side of (3.13) is positive. Suppose that (A, S) or (B, T) satisfies property (E.A), one of $A(X), B(X), S(X), T(X)$ is a closed subspace of X and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X .

Now, we prove a common fixed point Theorem of Gregus type using a strict contraction of integral type and a common property (E.A).

Theorem 5. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying (3.11) for all x, y in X for which the right-hand side of (3.11) is positive, where $0 < a < 1$ and ψ satisfies (3.3). Suppose that (A, S) and (B, T) satisfy a common property (E.A), $S(X)$ and $T(X)$ are closed subspaces of X and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X .

Proof. Suppose that (A, S) and (B, T) satisfy a common property (E.A). Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = z$ for some $z \in X$. Assume that $S(X)$ and $T(X)$ are closed subspaces of X . Then, $z = Su = Tv$ for some $u, v \in X$. If $Au \neq z$, using (3.11) we get

$$\begin{aligned} \int_0^{d(Au, By_n)} \psi(t) dt &< a \int_0^{d(Su, Ty_n)} \psi(t) dt + (1-a) \max \left\{ \int_0^{d(Au, Su)} \psi(t) dt, \right. \\ &\quad \int_0^{d(By_n, Ty_n)} \psi(t) dt, \left(\int_0^{d(Au, Su)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Ty_n)} \psi(t) dt \right)^{\frac{1}{2}}, \\ &\quad \left. \left(\int_0^{d(Su, By_n)} \psi(t) dt \right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au, Ty_n)} \psi(t) dt \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain

$$\int_0^{d(Au, z)} \psi(t) dt \leq (1-a) \int_0^{d(Au, z)} \psi(t) dt < \int_0^{d(Au, z)} \psi(t) dt$$

which is a contradiction. Then, $z = Au = Su$.

The rest of the proof follows as in Theorem 4. \square

If $B = A$ and $T = S$ in Theorem 5, we get the following corollary.

Corollary 5. Let A and S be mappings from a metric space (X, d) into itself satisfying (3.12) for all x, y in X for which the right-hand side of (3.12) is positive, where $0 < a < 1$ and ψ satisfies (3.3). Suppose that (A, S) satisfies property (E.A), $S(X)$ is a closed subspace of X and the pair (A, S) is weakly compatible. Then, A and S have a unique common fixed point in X .

If $\psi(t) = 1$ in Theorem 5, we get the following corollary.

Corollary 6. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying (3.13) for all x, y in X for which the right-hand side of (3.13) is positive, where $0 < a < 1$. Suppose that (A, S) and (B, T) satisfy common property (E.A), $S(X)$ and $T(X)$ are closed subspaces of X and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X .

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